

UNIVERSITY OF COPENHAGEN

Operations Research 3: Hierarchical Optimization and Equilibrium

by

Block 4, Year 2015/16

What?

1. Introduction
2. Optimality
3. Complementarity
4. Equilibrium
5. MPEC
6. EPEC

When?

- Week 17: 26 April, 29 April
- Week 18: 3 May, 6 May
- Week 19: 10 May, 13 May
- Week 20: 17 May, 20 May
- Week 21: 24 May, 27 May
- Week 22: 31 May, 3 June
- Week 23: 7 June, 10 June
- Week 24: 14 June, 17 June (tentative)

How?

- Theory, exercises and applications in all classes (using GAMS)
- Material provided for each block together with bibliography
- Active participation of students is expected (Socratic)
- Website, Absalon, discussion forum

Evaluation

- 30 minutes oral exam (Week 25: 23 June, 24 June)
- Approval of two projects:
 - Project A: Students must hand in some key exercises for each block. Each exercise must include nomenclature, description of the problem, formulation, data, results and conclusions, as well as the GAMS code.
 - Project B: Students must hand in a project of 5-10 pages in which a modified version of one of the models discussed in class is described, formulated and solved by GAMS.

1 Introduction

 Name

Learning objectives

- Define and identify convex sets
- Define and identify convex functions
- Define the main elements of an optimization problem
- Formulate a generic optimization problem
- Differentiate between local and global optima
- Identify optimization problems in which the global optima can be computed
- Formulate simple optimization problems
- Solve simple optimization problems using GAMS
- Enumerate the special features of electricity as a commodity

Bibliography

- Basic reading:
 - Boyd, Stephen P., and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004 (chapters 2, 3, 4).
 - Gomez-Exposito, Antonio, Antonio J. Conejo, and Claudio Cañizares, eds. Electric energy systems: analysis and operation. CRC Press, 2008 (chapters 1 y 5).
 - A GAMS tutorial
 - GAMS - A user guide

Definition of a convex set

A set $\mathcal{C}$ is convex if the line segment between any two points in $\mathcal{C}$ lies in $\mathcal{C}$.

$$x_1 \in \mathcal{C}$$

$$x_2 \in \mathcal{C}$$

$$0 \leq \theta \leq 1$$

$$\mathcal{C} \text{ is convex} \Leftrightarrow \theta x_1 + (1 - \theta)x_2 \in \mathcal{C}$$

Draw a convex set in $\mathbb{R}^2$

Draw a non-convex set in $\mathbb{R}^2$

Example 1-1: Is the intersection of two convex sets a convex set?

Definition of line

Let $x_1, x_2 \in \mathbb{R}^n$ be two points with $x_1 \neq x_2$, the line passing through x_1 and x_2 is given by $\{x \in \mathbb{R}^n : x = \theta x_1 + (1 - \theta)x_2\}$ with $\theta \in \mathbb{R}$.

Definition of line segment

Let $x_1, x_2 \in \mathbb{R}^n$ be two points with $x_1 \neq x_2$, the line segment between x_1 and x_2 is given by $\{x \in \mathbb{R}^n : x = \theta x_1 + (1 - \theta)x_2\}$ with $0 \leq \theta \leq 1$.

Example 1-2: Is a line a convex set? and a line segment?

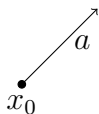
Definition of hyperplane

Let $a \in \mathbb{R}^n, a \neq 0$ and $x_0 \in \mathbb{R}^n$, then an hyperplane is a set of the form $\{x \in \mathbb{R}^n : a^T(x - x_0) = 0\}$.

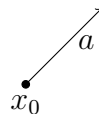
Definition of halfspace

Let $a \in \mathbb{R}^n, a \neq 0$ and $x_0 \in \mathbb{R}^n$, then a halfspace is a set of the form $\{x \in \mathbb{R}^n : a^T(x - x_0) \leq 0\}$.

Draw the hyperplane given a and x_0



Draw the halfspace given a and x_0



Example 1-3: Is an hyperplane a convex set?

📌 Example 1-4: Is a halfspace a convex set?

📖 Definition of polyhedron

A polyhedron is defined as the intersection of a finite number of halfspaces and hyperplanes, i.e., a set of the form $\{x \in \mathbb{R}^n : a_j^T x \leq b_j \ j = 1, \dots, m, c_j^T x = d_j \ j = 1, \dots, p\}$.

📌 Example 1-5: Is a polyhedron a convex set?

Definition of norm

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\text{dom} f = \mathbb{R}^n$ is a norm if:

- f is nonnegative: $f(x) \geq 0, \forall x \in \mathbb{R}^n$
- f is definite: $f(x) = 0 \Leftrightarrow x = 0$
- f is homogeneous: $f(tx) = |t|f(x), \forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y), \forall x, y \in \mathbb{R}^n$

Definition of norm ball

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a norm. A norm ball of radius r and center x_c is given by $\{x \in \mathbb{R}^n : f(x - x_c) \leq r\}$.

Example 1-6: Is a norm ball a convex set?

Ways to check whether a set is convex

- Use the formal definition
- Use the properties of convex sets (intersection,...)
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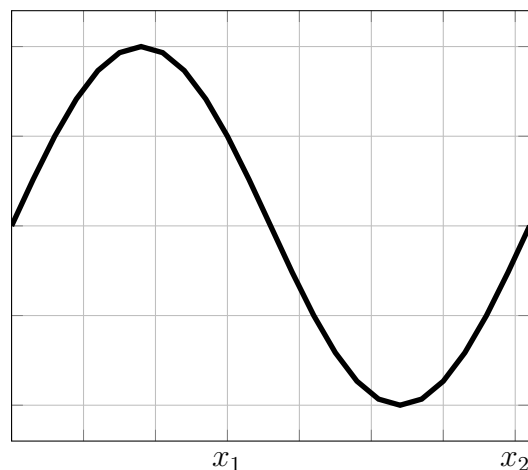
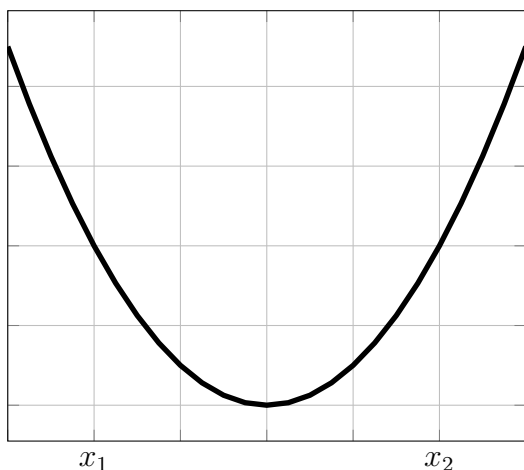
Definition of a convex function (I)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom} f$ is a convex set and if for all $x_1, x_2 \in \text{dom} f$ and $0 \leq \theta \leq 1$, then

$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$$

Graphical interpretation

- For each figure draw the point $(\theta x_1 + (1 - \theta)x_2, \theta f(x_1) + (1 - \theta)f(x_2))$ for $\theta = 0.5$
- For each figure draw the point $(\theta x_1 + (1 - \theta)x_2, f(\theta x_1 + (1 - \theta)x_2))$ for $\theta = 0.5$
- Do the same for $\theta = 0.25$ and $\theta = 0.75$



- Are these functions convex?
- What is the graphical interpretation of this definition of a convex function in $\mathbb{R}^2$?

 **Example 1-7:** Prove that a norm is a convex function in $\mathbb{R}^n$ using the definition above

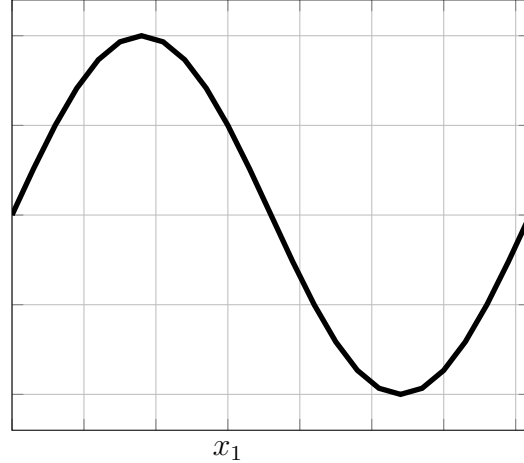
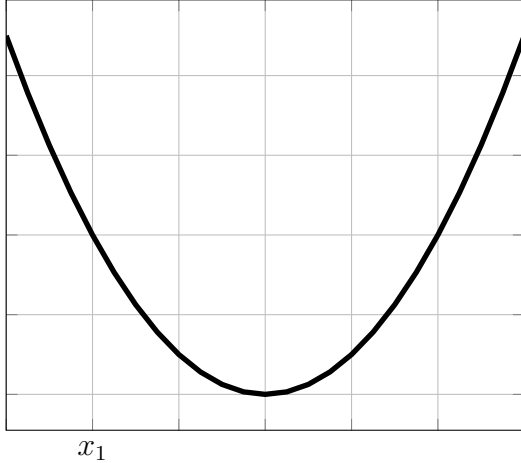
Definition of a convex function (II)

A differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom} f$ is a convex set and if

$$f(y) \geq f(x) + \nabla f(x)^T(y - x), \forall x, y \in \text{dom} f$$

Graphical interpretation

- For each figure draw the point the function $g(x) = f(x_1) + \nabla f(x_1)^T(x - x_1)$
- Compare $f(x)$ and $g(x)$ in the domain. Are these functions convex?



Proof of equivalence between definitions of convex functions

Let us assume a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $\text{dom} f$ a convex set. Let us assume that f is convex and that $x_1, x_2 \in \text{dom} f$. Since $\text{dom} f$ is a convex set, $x_1 + t(x_2 - x_1) \in \text{dom} f$ with $0 < t \leq 1$. Since f is convex we have that

$$f(x_1 + t(x_2 - x_1)) \leq$$

We divide both sides by t and obtain

$$f(x_2) \geq$$

Now we take the limit for $t \rightarrow 0$ and get

$$f(x_2) \geq$$

To prove it in the other direction take $x_1, x_2 \in \text{dom} f$ with $x_1 \neq x_2$, $0 \leq \theta \leq 1$ and $x_3 = \theta x_1 + (1 - \theta)x_2$. We assume that definition II holds and apply it twice as follows

$$f(x_1) \geq f(x_3) +$$

$$f(x_2) \geq f(x_3) +$$

We multiply the first inequality by θ and the second by $1 - \theta$ and add them to obtain

$$f(x_3) \leq$$

which proves that f is convex.

Definition of a convex function (III)

A twice differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{dom} f$ is a convex set and if

$$\nabla^2 f(x) \succeq 0, \forall x \in \text{dom} f$$

where $\succeq$ denotes a positive semidefinite matrix.

Definition of positive semidefinite matrix

A symmetric matrix $M \in \mathbb{R}^{n \times n}$ is positive semidefinite if

$$z^T M z \geq 0, \forall z \in \mathbb{R}^n$$

and we denote it as $M \in \mathbb{S}^n$. A matrix is positive semidefinite also if:

- All eigenvalues are non-negative
- Its leading principal minors are all non-negative

 **Example 1-8:** Establish under which conditions the function $f(x)$ defined as $\frac{1}{2}x^T P x + q^T x + r$ is convex in $\mathbb{R}^n$

 **Example 1-9:** Establish the values of parameter a for which the function $f(x)$ defined as $x^2 + y^2 + axy$ is convex in $\mathbb{R}^2$

Main elements and definitions of an optimization problem

An optimization problem is usually formulated as follows:

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

where $f_0(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function, $f_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ are the inequality constraints and $h_i(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ the equality constraints. Some definitions of an optimization problem are:

- *Domain*: The domain($\mathcal{D}$) is the intersection of the domains of $f_i(x)$ and $h_i(x)$.
- *Feasible point*: A point $\tilde{x} \in \mathcal{D}$ is feasible if $f_i(\tilde{x}) \leq 0, i = 1, \dots, m$ and $h_i(\tilde{x}) = 0, i = 1, \dots, p$.
- *Optimal value*: The optimal value z^* is defined as $z^* = \inf\{f_0(x) | f_i(\tilde{x}) \leq 0, i = 1, \dots, m, h_i(\tilde{x}) = 0, i = 1, \dots, p\}$.
- *Optimal point*: A point $\tilde{x} \in \mathcal{D}$ is optimal if $f_0(\tilde{x}) = z^*$. The optimal is usually denoted as x^* .
- *Locally optimal point*: A point $\tilde{x} \in \mathcal{D}$ is locally optimal if $f_0(\tilde{x}) = \inf\{f_0(x) | f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p, \|x - \tilde{x}\|_2 \leq R\}$ with $R > 0$.

Definition of convex optimization problem

An optimization problem in the form:

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

is convex if $f_0(x)$ and $f_i(x)$ are convex functions and $h_i(x)$ are affine functions.

Example 1-10: Are linear programming problems convex?

A general linear programming problem is formulated as

$$\begin{aligned} & \text{minimize} && c^T x + d \\ & \text{subject to} && Gx \preceq d \\ & && Ax = b \end{aligned}$$

with $x \in \mathbb{R}^n, d \in \mathbb{R}, c \in \mathbb{R}^n, G \in \mathbb{R}^{m \times n}, d \in \mathbb{R}^m, A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$.

Example 1-11: Are quadratic programming problems convex?

A general quadratic programming problem is formulated as

$$\begin{aligned} & \text{minimize} && x^T P x + q^T x + r \\ & \text{subject to} && Gx \preceq d \\ & && Ax = b \end{aligned}$$

with $x \in \mathbb{R}^n, r \in \mathbb{R}, q \in \mathbb{R}^n, P \in \mathbb{S}^n, G \in \mathbb{R}^{m \times n}, d \in \mathbb{R}^m, A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$.

✦ Example 1-12: Is the optimization problem below convex? Can you find an equivalent convex optimization problem?

$$\begin{aligned} & \text{minimize} && x_1^2 + x_2^2 \\ & \text{subject to} && \frac{x_1}{1 + x_2^2} \leq 0 \\ & && (x_1 + x_2)^2 = 0 \end{aligned}$$

Local and global optima

The most important property of convex optimization problems is that a locally optimal point is also a global optimal point. To prove this let us suppose that x_1 is feasible and locally optimal for a convex optimization problem and therefore it satisfies that

$$f_0(x_1) = \inf\{f_0(z) \mid f_i(z) \leq 0, i = 1, \dots, m, h_i(z) = 0, i = 1, \dots, p, \|z - x_1\|_2 \leq R\}, \quad R > 0 \quad (1)$$

Let us also suppose that x_1 is not global optimal, which means that there exists x_2 feasible such that

$$<$$

Let us now consider the point x_3 given by:

$$x_3 = (1 - \theta)x_1 + \theta x_2 \text{ with } \theta = \frac{R}{2\|x_2 - x_1\|_2}$$

Note that $\|x_3 - x_1\|_2 \leq R$. We know that x_3 is feasible because

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-

Since the optimization problem is convex we have that

$$f_0(x_3) \leq \quad \implies f_0(x_3) < f_0(x_1)$$

This contradicts (1) because $\|x_3 - x_1\|_2 \leq R$ but $f_0(x_3) < f_0(x_1)$ and therefore, there exists no feasible x_2 with $f_0(x_2) < f_0(x_1)$. Hence x_1 is global optimal.

📌 Example 1-13: Generation scheduling

We consider a generating unit with whose production cost can be expressed as $\mathcal{C}(x) = ax^2 + bx + c$, where x represents the production level at MWh, and a , b and c are known parameters. The generating unit has a maximum and minimum output levels of $\bar{x}$ and $\underline{x}$, respectively. If the electricity price is denoted by p , formulate an optimization problem that determines the optimal generation level to maximize the profit made by the generating unit. Is that optimization problem convex?

📁 Solving Example 1-13 with GAMS (Scheduling.gms)

Formulate Example 1-13 in GAMS and solve it for different price values using the following data: $\underline{x} = 0$, $\bar{x} = 100$, $a = 0.2$, $b = 10$, $c = 100$. Fill in the following table with the obtained results and comment on them.

p (€/MWh)	10	20	30	40	50
x (MWh)					

Example 1-14: Economic dispatch

We consider an economic dispatch problem (ED) in which the production schedules ($x_g, g = 1, \dots, N^G$) of different generating units are determined to satisfy a given demand (d) at the minimum cost. As in Example 1-13, the output of each unit is constrained by $\bar{x}_g$ and $\underline{x}_g$, and its operation cost is computed by the function $\mathcal{C}_g(x_g) = a_g x_g^2 + b_g x_g + c_g$. Formulate an optimization problem corresponding to an economic dispatch. Which kind of problem is this?

Solving Example 1-14 with GAMS (EcoDispatch1.gms)

Formulate Example 1-14 in GAMS and solve it for different demand values using the following data:

Unit	$\underline{x}_g$	$\bar{x}_g$	a_g	b_g	c_g
g_1	0	250	0.01	5	100
g_2	0	150	0.02	10	100
g_3	0	100	0.03	16	100

Fill in the following table with the obtained results and comment on them.

d (MWh)	100	200	300	400	500
x_{g1} (MWh)					
x_{g2} (MWh)					
x_{g3} (MWh)					

📌 Example 1-15: Economic dispatch (with network)

In Example 1-14 all generating units and loads are assumed to be located at the same node. However, electricity and other commodities are generally produced far from the consumption center and therefore, a network is required for its transportation. In this example, apart from the generating units and the demand, we also consider a number of nodes or buses ($n = 1, \dots, N^B$) and transmission lines ($l = 1, \dots, N^L$). The parameter h_{gn} is a unit-bus indicator equal to 1 if unit g is located at bus n , and 0 otherwise. The demand at each node is d_n , the flow through line l is denoted by f_l (with a maximum value of $\bar{f}_l$), and i_{ln} is an line-bus indicator parameter equal to 1/-1 for the sending/receiving bus of line l and to 0 otherwise. Formulate the economic dispatch considering network.

📁 Solving Example 1-15 with GAMS (EcoDispatch2.gms)

Formulate Example 1-15 in GAMS and solve it using the following data of a 3-bus system:

Unit	Bus	$\underline{x}_g$	$\bar{x}_g$	a_g	b_g	c_g	Bus	d_n	Line	$\bar{f}_l$
g_1	n_1	0	250	0.01	5	100	n_1	0	$l_1(n_1 - n_2)$	50
g_2	n_2	0	250	0.02	10	100	n_2	0	$l_2(n_1 - n_3)$	100
g_3	n_3	0	250	0.03	16	100	n_3	250	$l_3(n_2 - n_3)$	100

Fill in the following table with the obtained results and comment on them.

Unit	x_g	Line	f_l	Cost
g_1		l_1		
g_2		l_2		
g_3		l_3		



Discuss whether the model in Example 1-15 could be used for other commodities. What differentiates electricity from other commodities?



Example 1-16: Economic dispatch (with DC-network)

Modify the formulation of Example 1-15 knowing that in electricity network, the flow between two nodes is defined as:

$$f_l = B_l \sum_n i_{ln} \delta_n$$

where B_l is the line susceptance, and δ_n is the voltage angle at bus n .

 Solving Example 1-16 with GAMS (EcoDispatch3.gms)

Formulate Example 1-16 in GAMS and solve it using the following data:

Unit	Bus	$\underline{x}_g$	$\bar{x}_g$	a_g	b_g	c_g	Bus	d_n	Line	B_l	$\bar{f}_l$
g_1	n_1	0	250	0.01	5	100	n_1	0	$l_1(n_1 - n_2)$	100	50
g_2	n_2	0	250	0.02	10	100	n_2	0	$l_2(n_1 - n_3)$	100	100
g_3	n_3	0	250	0.03	16	100	n_3	250	$l_3(n_2 - n_3)$	100	100

Fill in the following table with the obtained results and compare them with those corresponding to Example 1-15. Discuss why the results are different.

Unit	x_g	Line	f_l	Cost
$g1$		l_1		
$g2$		l_2		
$g3$		l_3		